

MAS212 Linear Algebra I 2007-2008

Coursework 5 — Solutions

Exercise* 1. Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space with

$$\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{v}. \quad (1)$$

By (A4) there is an element $-\mathbf{v}$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Thus

$$\begin{aligned} \mathbf{u} &\stackrel{(A3)}{=} \mathbf{u} + \mathbf{0} \stackrel{(A4)}{=} \mathbf{u} + (\mathbf{v} + (-\mathbf{v})) \stackrel{(A2)}{=} (\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) \\ &\stackrel{(1)}{=} (\mathbf{w} + \mathbf{v}) + (-\mathbf{v}) \stackrel{(A2)}{=} \mathbf{w} + (\mathbf{v} + (-\mathbf{v})) \stackrel{(A4)}{=} \mathbf{w} + \mathbf{0} \stackrel{(A3)}{=} \mathbf{w}. \end{aligned}$$

Exercise* 2.

- (a) $S = \{ (1, s, t)^T \mid s, t \in \mathbb{R} \}$ is not a subspace of $\mathbb{R}^3$, since every subspace must contain the zero vector, but $(0, 0, 0) \notin S$.
- (b) $S = \{ (r, s, t)^T \mid r, s, t \in \mathbb{R} \text{ and } r + 2s - t = 0 \}$ is a subspace of $\mathbb{R}^3$. To see this note that $(0, 0, 0)^T \in S$, so S is not empty. Moreover, if $\mathbf{x}_1 = (r_1, s_1, t_1)^T \in S$ and $\mathbf{x}_2 = (r_2, s_2, t_2)^T \in S$ then $r_1 + 2s_1 - t_1 = 0$ and $r_2 + 2s_2 - t_2 = 0$, so

$$\mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} r_1 + r_2 \\ s_1 + s_2 \\ t_1 + t_2 \end{pmatrix} \in S,$$

since $(r_1 + r_2) + 2(s_1 + s_2) - (t_1 + t_2) = (r_1 + 2s_1 - t_1) + (r_2 + 2s_2 - t_2) = 0 + 0 = 0$. Furthermore, if $\mathbf{x} = (r, s, t)^T \in S$ and α is a scalar, then $r + 2s - t = 0$ and

$$\alpha \mathbf{x} = \begin{pmatrix} \alpha r \\ \alpha s \\ \alpha t \end{pmatrix} \in S$$

since $\alpha r + 2\alpha s - \alpha t = \alpha(r + 2s - t) = 0$.

- (c) $S = \{ (r, s, t)^T \mid r, s, t \in \mathbb{R} \text{ and } s^2 + t^2 \leq 1 \}$ is not a subspace of $\mathbb{R}^3$, since, for example, $\mathbf{x} = (0, 0, 1)^T \in S$, but $2\mathbf{x} = (0, 0, 2)^T \notin S$.

Exercise* 3.

- (a) True. Let $S = \{ A \in \mathbb{R}^{n \times n} \mid A \text{ is symmetric} \}$. Clearly, $O \in S$, so S is not empty. Let now A and B belong to S and let α be a real number. Then $A^T = A$ and $B^T = B$, and

$$(A + B)^T = A^T + B^T = A + B,$$

so $A + B \in S$. Furthermore

$$(\alpha A)^T = \alpha A^T = \alpha A,$$

so $\alpha A \in S$. Thus S is a subspace of $\mathbb{R}^{n \times n}$ as claimed.

- (b) False. Let $S = \{ A \in \mathbb{R}^{2 \times 2} \mid A \text{ is singular} \}$. Then S is not a subspace of $\mathbb{R}^{2 \times 2}$, since, for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S,$$

because $\det(A) = \det(B) = 0$, but $A + B \notin S$, since $\det(A + B) = \det(I) = 1 \neq 0$.

- (c) True. Let $S = \{f \in C[-1, 1] \mid f(-1) + f(1) = 0\}$. The zero function belongs to S , so S is not empty. Let now $f, g \in S$ and α be a real number. Then $f(-1) + f(1) = 0$ and $g(-1) + g(1) = 0$, so

$$(f+g)(-1) + (f+g)(1) = f(-1) + g(-1) + f(1) + g(1) = f(-1) + f(1) + g(-1) + g(1) = 0 + 0 = 0,$$

so $f + g \in S$. Moreover,

$$(\alpha f)(-1) + (\alpha f)(1) = \alpha f(-1) + \alpha f(1) = \alpha(f(-1) + f(1)) = 0,$$

so $\alpha f \in S$. Thus S is a subspace of $C[-1, 1]$.

- (d) False. The set $\{p \in P_7 \mid p \text{ has degree greater than } 4\}$ is not a subspace of P_7 , since it does not contain the zero polynomial.

Exercise* 4. Note that $\det(A) = 5 \neq 0$, so A is invertible. Thus, by the Invertible Matrix Theorem, the only solution of $Ax = 0$ is $x = 0$, so $N(A) = \{0\}$.

In order to determine the nullspace of B bring the corresponding augmented matrix to echelon form:

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -2 & 0 \\ 2 & 4 & -5 & 1 & 0 \\ -3 & -6 & 8 & 1 & 0 \end{array} \right) \xrightarrow[R_3 + 3R_1]{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & -3 & -2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & -1 & -5 & 0 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & -2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The free variables are x_2 and x_4 , so set $x_2 = \alpha$ and $x_4 = \beta$. Then $x_3 = -5x_4 = -5\beta$ and $x_1 = -2x_2 + 3x_3 + 2x_4 = -2\alpha - 15\beta + 2\beta = -2\alpha - 13\beta$, so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha - 13\beta \\ \alpha \\ -5\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -13 \\ 0 \\ -5 \\ 1 \end{pmatrix},$$

that is,

$$N(A) = \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -13 \\ 0 \\ -5 \\ 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

Exercise 5. Clearly, $O \in S$, since $FO = O$, so S is not-empty. Let $A, B \in S$ and let α be a scalar. Then $FA = O$ and $FB = O$, and it follows that

$$F(A + B) = FA + FB = O + O = O,$$

$$F(\alpha A) = \alpha FA = \alpha O = O.$$

Thus $A + B \in S$ and $\alpha A \in S$, that is, S is a subspace of $\mathbb{R}^{2 \times 4}$.

Exercise 6. First we show that $U \cap V$ is a subspace of W . Note that since neither U nor V are empty, they must both contain 0 . Thus $0 \in U \cap V$, hence $U \cap V$ is not empty. Let $w_1, w_2 \in U \cap V$ and let α be a scalar. Then w_1 and w_2 belong to both U and V , so $w_1 + w_2$ and αw_1 must belong to both U and V (since both U and V are subspaces). Thus $w_1 + w_2 \in U \cap V$ and $\alpha w_1 \in U \cap V$, that is, the intersection $U \cap V$ is a subspace of W .

In order to see that $U + V$ is a subspace of W , we first note that $0 \in U + V$ (0 belongs to both U and V , so $0 = 0 + 0 \in U + V$), thus $U + V$ is not empty. Let now $w_1, w_2 \in U + V$ and let α be a scalar. Then there are vectors $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that

$$w_1 = u_1 + v_1,$$

$$w_2 = u_2 + v_2.$$

Thus, by axioms (A1) and (A2),

$$\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \in U + V,$$

since $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{v}_1 + \mathbf{v}_2 \in V$. Moreover, by axiom (A5),

$$\alpha \mathbf{w}_1 = \alpha(\mathbf{u}_1 + \mathbf{v}_1) = \alpha \mathbf{u}_1 + \alpha \mathbf{v}_1 \in U + V,$$

since $\alpha \mathbf{u}_1 \in U$ and $\alpha \mathbf{v}_1 \in V$. Thus, the sum $U + V$ is a subspace of W .

Exercise* 7.

- (a) True. $\{(1, 0, 0)^T, (3, 1, 1)^T, (0, 1, 2)^T\}$ is a spanning set for $\mathbb{R}^3$. In order to see this, let $(a, b, c)^T$ be an arbitrary vector in $\mathbb{R}^3$. We need to find $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

that is, the α_i have to satisfy the system

$$\begin{array}{rcl} \alpha_1 + 3\alpha_2 & = & a \\ \alpha_2 + \alpha_3 & = & b \\ \alpha_2 + 2\alpha_3 & = & c \end{array}$$

Using Gaussian elimination, this system is equivalent to the system

$$\begin{array}{rcl} \alpha_1 + 3\alpha_2 & = & a \\ \alpha_2 + \alpha_3 & = & b \\ \alpha_3 & = & c - b \end{array}$$

which has the solution $\alpha_3 = c - b$, $\alpha_2 = b - \alpha_3 = 2b - c$, $\alpha_1 = a - 3\alpha_2 = a - 6b + 3c$, so

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a - 6b + 3c) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (2b - c) \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + (c - b) \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

that is, the set is a spanning set for $\mathbb{R}^3$ as claimed.

- (b) True. $\{(1, 0, 0)^T, (3, 1, 1)^T, (0, 1, 2)^T, (-1, 3, 4)^T\}$ is a spanning set for $\mathbb{R}^3$, because by (b) it contains vectors which already are a spanning set for $\mathbb{R}^3$.

- (c) True. $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a spanning set for $\mathbb{R}^{2 \times 2}$, since if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary vector in $\mathbb{R}^{2 \times 2}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}.$$

- (d) False. In order to see that $\{\mathbf{p}_1, \mathbf{p}_2\}$ is not a spanning set for P_2 , let $\mathbf{p}$ be an arbitrary polynomial in P_2 , say, $\mathbf{p}(x) = a + bx + cx^2$. If $\{\mathbf{p}_1, \mathbf{p}_2\}$ was a spanning set for P_2 , there would be real numbers α_1 and α_2 such that

$$\alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 = \mathbf{p},$$

that is

$$\alpha_1(2 + x) + \alpha_2 x^2 = a + bx + cx^2.$$

Comparing coefficients shows that α_1 and α_2 would have to satisfy

$$\begin{array}{rcl} 2\alpha_1 & = & a \\ \alpha_1 & = & b \\ \alpha_2 & = & c \end{array}$$

which is impossible if $a \neq 2b$.